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Yang-Mills Instantons in the Gravitational Instanton Backgrounds

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Abstract

The simplest and the most straightforward new algorithm for generating solutions to (anti) self-dual Yang-Mills (YM) equation in the typical gravitational instanton backgrounds is proposed. When applied to the Taub-NUT and the Eguchi-Hanson metrics, the two best-known gravitational instantons, the solutions turn out to be the rather exotic type of instanton configurations carrying finite YM action but generally fractional topological charge values.

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Well below the Planck scale, the strength of gravity is negligibly small relative to those of particle physics interactions described by non-abelian gauge theories. Nevertheless, as far as the topological aspect is concerned, gravity may have marked effects even at the level of elementary particle physics. Namely, the non-trivial topology of the gravitational field may play a role crucial enough to dictate the topological properties of, say, SU(2) Yang-Mills (YM) gauge field [1] as has been pointed out long ago [2]. Being an issue of great physical interest and importance, quite a few serious study along this line have appeared in the literature but they were restricted to the background gravitational field with high degree of isometry such as the Euclideanized Schwarzschild geometry [2] or the Euclidean de Sitter space [3]. Even the works involving more general background spacetimes including gravitational instantons (GI) were mainly confined to the case of asymptotically- locally-Euclidean (ALE) spaces which is one particular such GI and employed rather indirect and mathmatically-oriented solution generating methods such as the ADHM construction [11]. Here in this work we would like to propose a "simply physical" and hence perhaps the most direct algorithm for generating the YM instanton solutions in all species of known GI. And the essence of this method lies in writing the (anti) self-dual YM equation by employing truly relevant ansätz for the YM gauge connection and then directly solving it. To demonstrate how simple in method and powerful in applicability it is, we then apply this algorithm to the case of the Taub-NUT and the Eguchi-Hanson metrics, the two best-known GI. In particular, the actual YM instanton solution in the background of Taub-NUT metric (which is asymptotically-locally-flat (ALF) rather than ALE) is constructed for the first time in this work although its existence has been anticipated long ago in [2]. Interestingly, the solutions to (anti) self-dual YM equation turn out to be the rather exotic type of instanton configurations which are everywhere non-singular having *finite* YM action but sharing some features with meron solutions [9] such as their typical structure and generally *fractional* topological charge values carried by them. Namely, the YM instanton solution that we shall discuss in the background of GI in this work exhibit characteristics which are mixture of those of typical instanton and typical meron. This seems remarkable since it is well-known

that in flat spacetime, meron does not solve the 1st order (anti) self-dual equation although it does the 2nd order YM field equation and is singular at its center and has divergent action. In the loose sense, GI may be defined as a positive-definite metrics $g_{\mu\nu}$ on a complete and non-singular manifold satisfying the Euclidean Einstein equations and hence constituting the stationary points of the gravity action in Euclidean path integral for quantum gravity. But in the stricter sense [4], they are the metric solutions to the Euclidean Einstein equations having (anti) self-dual Riemann tensor

$$\tilde{R}_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ef} R_{efcd} = \pm R_{abcd} \tag{1}$$

(say, with indices written in non-coordinate orthonormal basis) and include only two families of solutions in a rigorous sense ; the Taub-NUT metric [5] and the Eguchi-Hanson instanton [6]. In the loose sense, however, there are several solutions to Euclidean Einstein equations that can fall into the category of GI. Thus we begin with the action governing our system, i.e., the Einstein-Yang-Mills (EYM) theory given by

$$I_{EYM} = \int_M d^4x\sqrt{g}\left[\frac{-1}{16\pi}R + \frac{1}{4g_c^2}F^a_{\mu\nu}F^{a\mu\nu}\right] - \int_{\partial M} d^3x\sqrt{h}\frac{1}{8\pi}K \tag{2}$$

where $F^a_{\mu\nu}$ is the field strength of the YM gauge field A^a_μ with $a = 1, 2, 3$ being the SU(2) group index and g_c being the gauge coupling constant. The Gibbons-Hawking term on the boundary ∂M of the manifold M is also added and h is the metric induced on ∂M and K is the trace of the second fundamental form on ∂M. Then by extremizing this action with respect to the metric $g_{\mu\nu}$ and the YM gauge field A^a_μ, one gets the following classical field equations respectively

$$\begin{aligned}
&R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu},\\
&T_{\mu\nu} = \frac{1}{g_c^2}\left[F^a_{\mu\alpha}F^{a\alpha}_{\nu} - \frac{1}{4}g_{\mu\nu}(F^a_{\alpha\beta}F^{a\alpha\beta})\right],\\
&D_\mu\left[\sqrt{g}F^{a\mu\nu}\right] = 0, \quad D_\mu\left[\sqrt{g}\tilde{F}^{a\mu\nu}\right] = 0
\end{aligned} \tag{3}$$

where we added Bianchi identity in the last line and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}A^b_\mu A^c_\nu$, $D^{ac}_\mu = \partial_\mu\delta^{ac} + \epsilon^{abc}A^b_\mu$ and $A_\mu = A^a_\mu(-iT^a)$, $F_{\mu\nu} = F^a_{\mu\nu}(-iT^a)$ with $T^a = \tau^a/2$ $(a = 1, 2, 3)$

being the SU(2) generators and finally $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}F_{\alpha\beta}$ is the (Hodge) dual of the field strength tensor. We now seek solutions $(g_{\mu\nu},\ A^a_\mu)$ of the coupled EYM equations given above in Euclidean signature obeying the (anti) self-dual equation in the YM sector

$$F^{\mu\nu} = g^{\mu\lambda}g^{\nu\sigma}F_{\lambda\sigma} = \pm\frac{1}{2}\epsilon_c^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{4}$$

where $\epsilon_c^{\mu\nu\alpha\beta} = \epsilon^{\mu\nu\alpha\beta}/\sqrt{g}$ is the curved spacetime version of totally antisymmetric tensor. As was noted in [2,3], in Euclidean signature, the YM energy-momentum tensor vanishes identically for YM fields satisfying this (anti) self-duality condition. This point is of central importance and can be illustrated briefly as follows. Under the Hodge dual transformation, $F^a_{\mu\nu} \to \tilde{F}^a_{\mu\nu}$, the YM energy-momentum tensor $T_{\mu\nu}$ given in eq.(3) above is invariant normally in Lorentzian signature. In Euclidean signature, however, its sign flips, i.e., $\tilde{T}_{\mu\nu} = -T_{\mu\nu}$. As a result, for YM fields satisfying the (anti) self-dual equation in Euclidean signature such as the instanton solution, $F^a_{\mu\nu} = \pm\tilde{F}^a_{\mu\nu}$, it follows that $T_{\mu\nu} = -\tilde{T}_{\mu\nu} = -T_{\mu\nu}$, namely the YM energy-momentum tensor vanishes identically, $T_{\mu\nu} = 0$. This, then, indicates that the YM field now does not disturb the geometry while the geometry still does have effects on the YM field. Consequently the geometry, which is left intact by the YM field, effectively serves as a "background" spacetime which can be chosen somewhat at our will (as long as it satisfies the vacuum Einstein equation $R_{\mu\nu} = 0$) and here in this work, we take it to be the gravitational instanton. Loosely speaking, all the typical GI, including Taub-NUT metric and Eguchi-Hanson solution, possess the same topology $R \times S^3$ and similar metric structures. Of course in a stricter sense, their exact topologies can be distinguished, say, by different Euler numbers and Hirzebruch signatures [4]. Particularly, in terms of the concise basis 1-forms, the metrics of these GI can be written as [4]

$$\begin{aligned} ds^2 &= c_r^2 dr^2 + c_1^2\left(\sigma_1^2 + \sigma_2^2\right) + c_3^2\sigma_3^2 \\ &= c_r^2 dr^2 + \sum_{a=1}^{3} c_a^2 \left(\sigma^a\right)^2 = e^A \otimes e^A \end{aligned} \tag{5}$$

where $c_r = c_r(r)$, $c_a = c_a(r)$, $c_1 = c_2 \neq c_3$ and the orthonormal basis 1-form e^A is given by

$$e^A = \left\{e^0 = c_r dr, \quad e^a = c_a\sigma^a\right\} \tag{6}$$

and $\{\sigma^a\}$ $(a = 1, 2, 3)$ are the left-invariant 1-forms satisfying the SU(2) Maurer-Cartan structure equation

$$d\sigma^a = -\frac{1}{2}\epsilon^{abc}\sigma^b \wedge \sigma^c. \tag{7}$$

They form a basis on the S^3 section of the geometry and hence can be represented in terms of 3-Euler angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, and $0 \leq \psi \leq 4\pi$ parametrizing S^3 as

$$\begin{aligned} \sigma^1 &= -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \\ \sigma^2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi, \\ \sigma^3 &= -d\psi - \cos\theta d\phi. \end{aligned} \tag{8}$$

Now in order to construct exact YM instanton solutions in the background of these GI, we now choose the relevant ansắtz for the YM gauge potential and the SU(2) gauge fixing. And in doing so, our general guideline is that the YM gauge field ansắtz should be endowed with the symmetry inherited from that of the background geometry, the GI. Thus we first ask what kind of isometry these GI possess. As noted above, typical GI, including the Taub-NUT and the Eguchi-Hanson metrics, possess the topology of $R \times S^3$. The geometrical structure of the S^3 section, however, is not that of perfectly "round" S^3 but rather, that of "squashed" S^3. In order to get a closer picture of this squashed S^3, we notice that the $r =$constant slices of these GI can be viewed as U(1) fibre bundles over $S^2 \sim CP^1$ with the line element

$$d\Omega_3^2 = c_1^2\left(\sigma_1^2 + \sigma_2^2\right) + c_3^2\sigma_3^2 = c_1^2 d\Omega_2^2 + c_3^2\left(d\psi + B\right)^2 \tag{9}$$

where $d\Omega_2^2 = (d\theta^2 + \sin^2\theta d\phi^2)$ is the metric on unit S^2, the base manifold whose volume form Ω_2 is given by $\Omega_2 = dB$ as $B = \cos\theta d\phi$ and ψ then is the coordinate on the U(1)$\sim S^1$ fibre manifold. Now then the fact that $c_1 = c_2 \neq c_3$ indicates that the geometry of this fibre bundle manifold is not that of round S^3 but that of squashed S^3 with the squashing factor given by (c_3/c_1). And further, it is squashed along the U(1) fibre direction. Thus this failure for the geometry to be that of exactly round S^3 keeps us from writing down

the associated ansätz for the YM gauge potential right away. Apparently, if the geometry were that of round S^3, one would write down the YM gauge field ansätz as $A^a = f(r)\sigma^a$ [3] with $\{\sigma^a\}$ being the left-invariant 1-forms introduced earlier. The rationale for this choice can be stated briefly as follows. First, since the r =constant sections of the background space have the geometry of round S^3 and hence possess the SO(4)-isometry, one would look for the SO(4)-invariant YM gauge connection ansätz as well. Next, noticing that both the r =constant sections of the frame manifold and the SU(2) YM group manifold possess the geometry of round S^3, one may naturally choose the left-invariant 1-forms $\{\sigma^a\}$ as the "common" basis for both manifolds. Thus this YM gauge connection ansätz, $A^a = f(r)\sigma^a$ can be thought of as a hedgehog-type ansätz where the group-frame index mixing is realized in a simple manner [3]. Then coming back to our present interest, namely the GI given in eq.(5), in r =constant sections, the SO(4)-isometry is partially broken down to that of SO(3) by the squashedness along the U(1) fibre direction to a degree set by the squashing factor (c_3/c_1). Thus now our task became clearer and it is how to encode into the YM gauge connection ansätz this particular type of SO(4)-isometry breaking coming from the squashed S^3. Interestingly, a clue to this puzzle can be drawn from the work of Eguchi and Hanson [7] in which they constructed abelian instanton solution in Euclidean Taub-NUT metric (namely the abelian gauge field with (anti)self-dual field strength with respect to this metric). To get right to the point, the working ansätz they employed for the abelian gauge field to yield (anti)self-dual field strength is to align the abelian gauge connection 1-form along the squashed direction, i.e., along the U(1) fibre direction, $A = g(r)\sigma^3$. This choice looks quite natural indeed. After all, realizing that embedding of a gauge field in a geometry with high degree of isometry is itself an isometry (more precisely isotropy)-breaking action, it would be natural to put it along the direction in which part of the isometry is already broken. Finally therefore, putting these two pieces of observations carefully together, now we are in the position to suggest the relevant ansätz for the YM gauge connection 1-form in these GI and it is

$$A^a = f(r)\sigma^a + g(r)\delta^{a3}\sigma^3 \tag{10}$$

which obviously would need no more explanatory comments except that in this choice of the ansätz, it is implicitly understood that the gauge fixing $A_r = 0$ is taken. From this point on, the construction of the YM instanton solutions by solving the (anti)self-dual equation given in eq.(4) is straightforward. To sketch briefly the computational algorithm, first we obtain the YM field strength 2-form (in orthonormal basis) via exterior calculus (since the YM gauge connection ansätz is given in left-invariant 1-forms) as $F^a = (F^1, F^2, F^3)$ where

$$\begin{aligned}
F^1 &= \frac{f'}{c_r c_1}(e^0 \wedge e^1) + \frac{f[(f-1)+g]}{c_2 c_3}(e^2 \wedge e^3), \\
F^2 &= \frac{f'}{c_r c_2}(e^0 \wedge e^2) + \frac{f[(f-1)+g]}{c_3 c_1}(e^3 \wedge e^1), \\
F^3 &= \frac{(f'+g')}{c_r c_3}(e^0 \wedge e^3) + \frac{[f(f-1)-g]}{c_1 c_2}(e^1 \wedge e^2)
\end{aligned} \tag{11}$$

from which we can read off the (anti)self-dual equation to be

$$\pm \frac{f'}{c_r c_1} = \frac{f[(f-1)+g]}{c_2 c_3}, \qquad \pm \frac{(f'+g')}{c_r c_3} = \frac{[f(f-1)-g]}{c_1 c_2} \tag{12}$$

where "+" for self-dual and "−" for anti-self-dual equation and we have only a set of two equations as $c_1 = c_2$. The specifics of different GI are characterized by particular choices of the orthonormal basis $e^A = \{e^0 = c_r dr, \;\; e^a = c_a \sigma^a\}$. Thus next, for each GI (i.e., for each choice of e^A), we solve the (anti)self-dual equation in (12) for ansätz functions $f(r)$ and $g(r)$ and finally from which the YM instanton solutions in eq.(10) and their (anti)self-dual field strength in eq.(11) can be obtained. We now present the solutions obtained by applying the algorithm presented here to the two best-known GI, the Taub-NUT and the Eguchi-Hanson metrics.

(I) YM instanton in Taub-NUT (TN) metric background

The TN GI solution written in the metric form given in eq.(5) amounts to

$$c_r = \frac{1}{2}\left[\frac{r+m}{r-m}\right]^{1/2}, \quad c_1 = c_2 = \frac{1}{2}\left[r^2 - m^2\right]^{1/2}, \quad c_3 = m\left[\frac{r-m}{r+m}\right]^{1/2}$$

and it is a solution to Euclidean vacuum Einstein equation $R_{\mu\nu} = 0$ for $r \geq m$ with self-dual Riemann tensor. The apparent singularity at $r = m$ can be removed by a coordinate

redefinition and is a 'nut' (in terminology of Gibbons and Hawking [4]) at which the isometry generated by the Killing vector $(\partial/\partial\psi)$ has a zero-dimensional fixed point set. And this TN instanton is an asymptotically-locally-flat (ALF) metric.

It turns out that only the anti-self-dual equation $F^a = -\tilde{F}^a$ admits a non-trivial solution and it is $A^a = (A^1, A^2, A^3)$ where

$$A^1 = \pm 2\frac{(r-m)^{1/2}}{(r+m)^{3/2}}e^1, \quad A^2 = \pm 2\frac{(r-m)^{1/2}}{(r+m)^{3/2}}e^2, \quad A^3 = \frac{(r+3m)}{m}\frac{(r-m)^{1/2}}{(r+m)^{3/2}}e^3 \tag{13}$$

and $F^a = (F^1, F^2, F^3)$ where

$$\begin{aligned} F^1 &= \pm\frac{8m}{(r+m)^3}\left(e^0\wedge e^1 - e^2\wedge e^3\right), \quad F^2 = \pm\frac{8m}{(r+m)^3}\left(e^0\wedge e^2 - e^3\wedge e^1\right), \\ F^3 &= \frac{16m}{(r+m)^3}\left(e^0\wedge e^3 - e^1\wedge e^2\right). \end{aligned} \tag{14}$$

It is interesting to note that this YM field strength and the Ricci tensor of the background TN GI are proportional as $|F^a| = 2|R^0_a|$ except for opposite self-duality, i.e.,

$$\begin{aligned} R^0_1 &= -R^2_3 = \frac{4m}{(r+m)^3}\left(e^0\wedge e^1 + e^2\wedge e^3\right), \quad R^0_2 = -R^3_1 = \frac{4m}{(r+m)^3}\left(e^0\wedge e^2 + e^3\wedge e^1\right), \\ R^0_3 &= -R^1_2 = -\frac{8m}{(r+m)^3}\left(e^0\wedge e^3 + e^1\wedge e^2\right). \end{aligned} \tag{15}$$

(II) YM instanton in Eguchi-Hanson (EH) metric background

The EH GI solution amounts to

$$c_r = \left[1-\left(\frac{a}{r}\right)^4\right]^{-1/2}, \quad c_1 = c_2 = \frac{1}{2}r, \quad c_3 = \frac{1}{2}r\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}$$

and again it is a solution to Euclidean vacuum Einstein equation $R_{\mu\nu} = 0$ for $r \geq a$ with self-dual Riemann tensor. $r = a$ is just a coordinate singularity that can be removed by a coordinate redefinition provided that now ψ is identified with period 2π rather than 4π and is a 'bolt' (in terminology of Gibbons and Hawking [4]) where the action of the Killing field $(\partial/\partial\psi)$ has a two-dimensional fixed point set. Besides, this EH instanton is an asymptotically-locally-Euclidean (ALE) metric.

In this time, only the self-dual equation $F^a = +\tilde{F}^a$ admits a non-trivial solution and it is $A^a = (A^1, A^2, A^3)$ where

$$A^1 = \pm\frac{2}{r}\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2} e^1, \quad A^2 = \pm\frac{2}{r}\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2} e^2, \quad A^3 = \frac{2}{r}\frac{\left[1+\left(\frac{a}{r}\right)^4\right]}{\left[1-\left(\frac{a}{r}\right)^4\right]^{1/2}} e^3 \tag{16}$$

and $F^a = (F^1, F^2, F^3)$ where

$$\begin{aligned}
F^1 &= \pm\frac{4}{r^2}\left(\frac{a}{r}\right)^4\left(e^0\wedge e^1 + e^2\wedge e^3\right), \quad F^2 = \pm\frac{4}{r^2}\left(\frac{a}{r}\right)^4\left(e^0\wedge e^2 + e^3\wedge e^1\right), \\
F^3 &= -\frac{8}{r^2}\left(\frac{a}{r}\right)^4\left(e^0\wedge e^3 + e^1\wedge e^2\right).
\end{aligned} \tag{17}$$

Again it is interesting to realize that this YM field strength and the Ricci tensor of the background EH GI are proportional as $|F^a| = 2|R^0_a|$, i.e.,

$$\begin{aligned}
&R^0_1 = -R^2_3 = \frac{2}{r^2}\left(\frac{a}{r}\right)^4\left(-e^0\wedge e^1 + e^2\wedge e^3\right), \quad R^0_2 = -R^3_1 = \frac{2}{r^2}\left(\frac{a}{r}\right)^4\left(-e^0\wedge e^2 + e^3\wedge e^1\right), \\
&R^0_3 = -R^1_2 = -\frac{4}{r^2}\left(\frac{a}{r}\right)^4\left(-e^0\wedge e^3 + e^1\wedge e^2\right).
\end{aligned} \tag{18}$$

It is also interesting to note that this YM instanton solution particularly in EH background (which is ALE) obtained by directly solving the self-dual equation can also be "constructed" by simply identifying $A^a = \pm 2\omega^0_a$ (where $\omega^0_a = (\epsilon_{abc}/2)\omega^{bc}$ are the spin connection of EH metric) and hence $F^a = \pm 2R^0_a$ as was noticed by [10] but in the string theory context with different motivation. This construction of solution via a simple identification of gauge field connection with the spin connection, however, works only in ALE backgrounds such as EH metric and generally fails as is manifest in the previous TN background case (which is ALF, not ALE) in which $A^a \neq \pm 2\omega^0_a$ but still $F^a = \pm 2R^0_a$. Thus the method presented here by first writing (by employing a relevant ansätz for the YM gauge connection given in eq.(10)) and directly solving the (anti) self-dual equation looks to be the algorithm for generating the solution with general applicability to all species of GI in a secure and straightforward manner. Indeed, the detailed and comprehensive coverage of YM instanton solutions in all other GI based on the algorithm presented in this work will be reported elsewhere and it will show how simple albeit powerful this method really is. In this regard, the method for generating YM instanton solutions to (anti) self-dual equation in all known GI backgrounds proposed here in this work can be contrasted to earlier works in the literature [12] discussing

the construction of YM instantons mainly in the background of ALE GI via indirect methods such as that of ADHM [11].

Having constructed explicit YM instanton solutions in TN and EH GI, we now turn to the physical interpretation of the structure of these SU(2) YM instantons supported by the two typical GI. Recall that the relevant ansätz for the YM gauge connection is of the form $A^a = f(r)\sigma^a$ in the background geometry such as de Sitter GI [3] with topology of $R \times (\text{round})S^3$ and of the form $A^a = f(r)\sigma^a + g(r)\delta^{a3}\sigma^3$ in the less symmetric GI backgrounds with topology of $R \times (\text{squashed})S^3$. Thus in order to get some insight into the physical meaning of the structure of these YM connection ansätz, we now try to re-express the left-invariant 1-forms $\{\sigma^a\}$ forming a basis on S^3 in terms of more familiar Cartesian coordinate basis. Utilizing the coordinate transformation from polar $(r,\ \theta,\ \phi,\ \psi)$ to Cartesian (t, x, y, z) coordinates (note, here, that t is not the usual "time" but just another spacelike coordinate) given by [4]

$$x + iy = r\cos\frac{\theta}{2}e^{\frac{i}{2}(\psi+\phi)}, \quad z + it = r\sin\frac{\theta}{2}e^{\frac{i}{2}(\psi-\phi)}, \tag{19}$$

where $x^2+y^2+z^2+t^2 = r^2$ and further introducing the so-called 'tHooft tensor [1,9] defined by $\eta^{a\mu\nu} = -\eta^{a\nu\mu} = (\epsilon^{0a\mu\nu} + \epsilon^{abc}\epsilon^{bc\mu\nu}/2)$, the left-invariant 1-forms can be cast to a more concise form $\sigma^a = 2\eta^a_{\mu\nu}(x^\nu/r^2)dx^\mu$. Therefore, the YM instanton solution, in Cartesian coordinate basis, can be written as

$$A^a = A^a_\mu dx^\mu = 2\left[f(r) + g(r)\delta^{a3}\right]\eta^a_{\mu\nu}\frac{x^\nu}{r^2}dx^\mu \tag{20}$$

in the background of TN and EH GI with topology of $R \times (\text{squashed})S^3$. Now to appreciate the meaning of this structure, we go back to the flat space situation. As is well-known, in flat space, the standard BPST [1] SU(2) YM instanton solution takes the form $A^a_\mu = 2\eta^a_{\mu\nu}[x^\nu/(r^2+\lambda^2)]$ with λ being the size of the instanton. Note, however, that separately from this BPST instanton solution, there is another non-trivial solution to the YM field equation of the form $A^a_\mu = \eta^a_{\mu\nu}(x^\nu/r^2)$ found long ago by De Alfaro, Fubini, and Furlan [8]. This second solution is called "meron" [9] as it carries a half unit of topological charge

and is known to play a certain role concerning the quark confinement [9]. It, however, exhibits singularity at its center $r = 0$ and hence has a diverging action and falls like $1/r$ as $r \to \infty$. Thus we are led to the conclusion that the YM instanton solution in typical GI backgrounds possess the structure of (curved space version of) meron at large r. As is well-known, in flat spacetime meron does not solve the 1st order (anti) self-dual equation although it does the second order YM field equation. Thus in this sense, this result seems remarkable since it implies that in the GI backgrounds, the (anti) self-dual YM equation admits solutions which exhibit the configuration of meron solution at large r in contrast to the flat spacetime case. And we only conjecture that when passing from the flat (R^4) to GI ($R \times S^3$) geometry, the closure of the topology of part of the manifold appears to turn the structure of the instanton solution from that of standard BPST into that of meron. Next, we look into the behavior of these solutions in TN and EH GI backgrounds as $r \to 0$. For TN and EH instantons, the ranges for radial coordinates are $m \leq r < \infty$ and $a \leq r < \infty$, respectively. Since the point $r = 0$ is absent in these manifolds, the solutions in these GI are everywhere regular. Finally, we close with perhaps the most interesting comments on the estimate of the instanton contribution to the intervacua tunnelling amplitude. It has been pointed out in the literature that both in the background of Euclidean Schwarzschild geometry [2] and in the Euclidean de Sitter space [3], the (anti) instanton solutions have the Pontryagin index of $\nu[A] = \pm 1$ and hence give the contribution to the (saddle point approximation to) intervacua tunnelling amplitude of $\exp\left[-8\pi^2/g_c^2\right]$, which, interestingly, are the same as their flat space counterparts even though these curved space YM instanton solutions do not correspond to gauge transformations of any flat space instanton solution [1]. This unexpected and hence rather curious property, however, turns out not to persist in YM instantons in GI backgrounds such as TN and EH metrics. In order to see this, consider the curved space version of Pontryagin index or second Chern class having the interpretation of instanton number $\nu[A]$ given by

$$\nu[A] = Ch_2(F) = \frac{-1}{8\pi^2}\int_{M^4} tr(F \wedge F) = \int_{R\times S^3} d^4x\sqrt{g}\left[\frac{-1}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\right] \tag{21}$$

and the saddle point approximation to the intervacua tunnelling amplitude

$$\Gamma_{GI} \sim \exp\left[-I_{GI}(instanton)\right] \tag{22}$$

where the subscript "GI" denotes corresponding quantities in the GI backgrounds and $I_{GI}(instanton)$ represents the Euclidean YM theory action evaluated at the YM instanton solution, i.e.,

$$I_{GI}(instanton) = \int_{R\times S^3} d^4x\sqrt{g}\left[\frac{1}{4g_c^2}F^a_{\mu\nu}F^{a\mu\nu}\right] = \left(\frac{8\pi^2}{g_c^2}\right)|\nu[A]| \tag{23}$$

where we used the (anti)self-duality relation $F^a = \pm\tilde{F}^a$. Then the straightforward calculation yields ; $\nu[A] = 1$, $I_{GI}(instanton) = 8\pi^2/g_c^2$ and $\Gamma_{GI} \sim \exp\left(-8\pi^2/g_c^2\right)$ for the instanton solution in TN metric and $\nu[A] = -3/2$, $I_{GI}(instanton) = 12\pi^2/g_c^2$ and $\Gamma_{GI} \sim \exp\left(-12\pi^2/g_c^2\right)$ for the instanton solution in EH metric background. Here, however, the solution in EH metric background carries the half-integer Pontryagin index actually because the boundary of EH space is S^3/Z_2 [13]. Therefore we need to be cautious in drawing the conclusion that the fact that solutions in GI backgrounds carry fractional topological charges could be another supporting evidence for meron interpretation of the solutions. To summarize, in the present work we constructed the solutions to (anti)self-dual YM equation in the typical gravitational instanton geometries and analyzed their physical nature. As demonstrated, the solutions turn out to take the structure of merons at large r and generally carry fractional topological charge values. Nevertheless, it seems more appropriate to conclude that the solutions still should be identified with (curved space version of) instantons as they are solutions to 1st order (anti) self-dual equation and are everywhere regular having finite YM action. However, these curious mixed characteristics of the solutions to (anti) self-dual YM equation in GI backgrounds appear to invite us to take them more seriously and further explore potentially interesting physics associated with them.

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